

# A Sturm comparison theorem for first order systems of differential equations

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## RESUMEN

En este trabajo se establece y demuestra un teorema de comparación de Sturm para sistemas de ecuaciones diferenciales de primer orden.

En general, un teorema de comparación establece que una solución de una ecuación diferencial, o sistema, tiene un cero en un dominio acotado, abierto, conexo, no vacío, o en ese dominio más su frontera, cuando una ecuación diferencial, o sistema, asociado al anterior, tiene una solución que se anula en la frontera de dicho dominio.

## INTRODUCTION

The first results on comparison and separation theorems for the first components of real solutions of systems of differential equations of the form

$$w' = Pw + Qz$$

$$z' = Rw + Sz$$

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were obtained by M. Bôcher [1] in 1902, by making use of the Riccati equation. E. Kamke [3], in 1939, obtained the same results using the Prüfer Transformation.

The comparison theorems of Bôcher and Kamke are results based on pointwise inequalities among the coefficients of the systems, they were not generalized until recently, 1971, by J.B. Díaz and J.R. McLaughlin [2], by means of an identity similar to Picone's [6].

The following comparison theorem is based in conditions of a mixed type, pointwise and integral inequalities, and generalizes the results in [4].

### THEOREM

Suppose that  $w_i, z_i, (i = 1, 2)$ , is a non-trivial real-valued solution in  $[a, b]$  of the system

$$\begin{pmatrix} w_i \\ z_i \end{pmatrix}' = \begin{pmatrix} P_i & Q_i \\ R_i & S_i \end{pmatrix} \begin{pmatrix} w_i \\ z_i \end{pmatrix}$$

where  $P_i, Q_i, R_i, S_i$ , are real-valued and continuous functions on  $[a, b]$ .

Let  $\sigma$  and  $\tau$  be defined by

$$\sigma w_1(a) - z_1(a) = 0$$

$$\tau w_2(a) - z_2(a) = 0,$$

and assume that

$$w_1(b) = 0, \quad w_1(x) \neq 0 \quad \text{in} \quad a \leq x < b.$$

Furthermore, suppose that

$$(1) \quad \left| \sigma + \int_a^x \left( R_1 + \frac{(S_1 - P_1)^2}{4 Q_1} \right) dt \right| \leq \tau + \int_a^x \left( R_2 + \frac{(S_2 - P_2)^2}{4 Q_2} \right) dt$$

for  $a \leq x \leq b$ , and that

$$(2) \quad \left\{ \begin{array}{l} Q_2(x) \leq Q_1(x) < 0, \\ \left| \frac{S_1 - P_1}{2 \sqrt{-Q_1}} \right| \leq \frac{S_2 - P_2}{2 \sqrt{-Q_2}}, \quad \text{for } a \leq x \leq b. \end{array} \right.$$

Then,  $w_2(x)$  has at least one zero in  $(a, b]$ .

PROOF

Assume first that strict inequality holds in (1). We define two new functions  $u_i$ ,  $(i = 1, 2)$ , by

$$u_i = \frac{z_i}{w_i} ;$$

they satisfy the Riccati equations,

$$(3)_i \quad u' = -Q_i u^2 + (S_i - P_i)u + R_i, \quad (i = 1, 2)$$

with initial conditions  $u_1(a) = \sigma$ ,  $u_2(a) = \tau$ .

Suppose that, contrary to the assertion of the theorem,  $w_2(x) \neq 0$  on  $[a, b]$ . Then,  $(3)_2$  makes sense for all  $a \leq x \leq b$ , and we have

$$(4) \quad u_2(x) = \tau + \int_a^x (-Q_2 u_2^2) dt + \int_a^x (S_2 - P_2) u_2 dt + \\ + \int_a^x R_2 dt = \tau + \int_a^x \left( \sqrt{-Q_2} u_2 + \frac{S_2 - P_2}{2\sqrt{-Q_2}} \right)^2 dt + \\ + \int_a^x \left( R_2 + \frac{(S_2 - P_2)^2}{4Q_2} \right) dt.$$

Since  $w_1(x) \neq 0$ ,  $a \leq x < b$ ,  $(3)_1$  has meaning for all  $x \in [a, c]$ , where  $a \leq c < b$ .

Therefore,  $u_1(x)$  is defined on  $[a, c]$  and in this interval,

$$(5) \quad u_1(x) = \sigma + \int_a^x \left( \sqrt{-Q_1} u_1 + \frac{S_1 - P_1}{2\sqrt{-Q_1}} \right)^2 dt + \\ + \int_a^x \left( R_1 + \frac{(S_1 - P_1)^2}{2Q_1} \right) dt.$$

We claim that  $|u_1(x)| < u_2(x)$  for all  $x$  in  $[a, c]$ ,  $c \in [a, b)$ .

Indeed, from (1), (4) and (5) we have,

$$\begin{aligned} u_2(x) &> \tau + \int_a^x \left( R_2 + \frac{(S_2 - P_2)^2}{4 Q_2} \right) dt > \\ &> -\left( \sigma + \int_a^x \left( R_1 + \frac{(S_1 - P_1)^2}{4 Q_1} \right) dt \right) \cong \\ &\cong -\left( \sigma + \int_a^x \left( R_1 + \frac{(S_1 - P_1)^2}{4 Q_1} \right) dt + \int_a^x \left( \sqrt{-Q_1} u_1 + \frac{S_1 - P_1}{2 \sqrt{-Q_1}} \right)^2 dt \right) = \\ &= -u_1(x), \end{aligned}$$

for all  $x$  in  $[a, c]$ .

Let

$$d = \inf \{ x \mid u_2(x) \leq u_1(x), a \leq x \leq b \}.$$

Assume  $d \leq c$ . (Otherwise the claim holds).

Since  $a < d$  by hypothesis, we have  $u_1(x) < u_2(x)$  for all  $a \leq x < d$ , and  $u_1(d) = u_2(d)$ .

Therefore,  $u_2(x) \geq |u_1(x)|$  in  $[a, d]$ , and

$$\begin{aligned} \sqrt{-Q_2} u_2 + \frac{S_2 - P_2}{2 \sqrt{-Q_2}} &\geq \left| \sqrt{-Q_1} u_1 \right| + \left| \frac{S_1 - P_1}{2 \sqrt{-Q_1}} \right| \geq \\ &\left| \sqrt{-Q_1} u_1 + \frac{S_1 - P_1}{2 \sqrt{-Q_1}} \right|, \end{aligned}$$

$a \leq x \leq d$ , by (2).

Thus,

$$u_2(d) = \tau + \int_a^d \left( \sqrt{-Q_2} u_2 + \frac{S_2 - P_2}{2 \sqrt{-Q_2}} \right)^2 dt + \int_a^d \left( R_2 + \frac{(S_2 - P_2)^2}{4 Q_2} \right) dt$$

$$+ \int_a^d \left( \sqrt{-Q_1} u_1 + \frac{S_1 - P_1}{2 \sqrt{-Q_1}} \right)^2 dt + \int_a^d \left( R_1 + \frac{(S_1 - P_1)^2}{4 Q_1} \right) dt = u_1(d),$$

contradicting the assumption  $u_1(d) = u_2(d)$ . This proves the inequality  $|u_1(x)| < u_2(x)$  for all  $x$  in the interval  $[a, c]$ , with  $a \leq c < b$ . The same argument used in [5] proves  $|u_1(x)| \leq u_2(x)$  in  $[a, c]$ ,  $c \in [a, b)$ , when we assume weak inequality in (1).

The fact that  $u_2$  is continuous in  $[a, b]$  and the above result imply the existence of a constant  $M$  such that

$$|u_1(x)| < M$$

for all  $a \leq x < b$ . This contradicts the hypothesis

$$\lim_{x \rightarrow b} u_1(x) = \infty$$

and proves that  $w_2(x)$  has at least one zero in  $(a, b]$ .



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